ON CANONICALLY DERIVED FAMILIES OF SURFACES OF GENERAL TYPE OVER CURVES

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The aim of this paper is to study a family of surfaces of general type over a curve which is derived from a multicanonical system of a smooth projective threefold of general type. As far as we know, there are no systematical references yet on this topic. In fact, to study this kind of families is an important step of the classification theory.

We always suppose that the ground field is algebraically closed of characteristic zero. Let X be a smooth projective variety of general type with dimension d. We say that $|mK_X|$ *is composed of a pencil of varieties of dimension* $d-1$ if $\dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(mK_X)) \geq 2$ and $\dim \phi_m(X) = 1$, where m is a positive integer and $\phi_m := \Phi_{|mK_X|}$ is the rational map defined by the system $|mK_X|$. Set $P_m(X) := \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(mK_X))$. We call $P_m(X)$ *the m-th genus of* X, which is an important birational invariant.

Now suppose $|mK_X|$ is composed of a pencil. Take possible blow-ups $\pi : X' \longrightarrow X$, according to Hironaka, such that $g_m := \phi_m \circ \pi$ is a morphism onto its image. Denote

$$W_m := \overline{\phi_m(X)} \subset \mathbb{P}^{P_m(X)-1}$$

and let

$$g_m : X' \xrightarrow{f_m} C \xrightarrow{\psi_m} W_m$$

be the Stein factorization of g_m, where C is a smooth projective curve of genus $b := g(C)$. Then we have the following commutative diagram:

$$\begin{array}{ccccc} X' & \xrightarrow{id.} & X' & \xrightarrow{\pi} & X \\ {\scriptstyle f_m}\downarrow & & \downarrow{\scriptstyle g_m} & & \downarrow{\scriptstyle \phi_m} \\ C & \xrightarrow[\psi_m]{} & W_m & \xleftarrow[id.]{} & W_m \end{array}$$

where we note that ϕ_m is only a rational map. Denote by F a general fiber of f_m. Then F is a smooth projective variety of general type of dimension $d-1$. We say

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that the fibration $f_m : X' \longrightarrow C$ is *a derived family from the m-canonical pencil* $|mK_X|$, which is the main object of this paper.

When $d = 2$ and $m = 1$, there are infinite number of such families according to [Be] and the classification is still incomplete according to [Ca2] where Catanese first constructed a canonically derived family of curves with non-constant moduli. When $d = 2$ and $m \geq 2$, only one possibility occurs according to Theorem 1 of [X1]. Explicitly, such a family of curves is derived from the bicanonical system of a smooth surface S of general type with the invariants $(K^2, p_g) = (1, 0)$, where K^2 and p_g are both the invariants of the minimal model of S. We note that, in this situation, $P_2(S) = 2$ and both $p_g(S)$ and $q(S)$ take minimal values.

It is natural we turn our interest to higher dimensional case. Here, we only treat the case $d = 3$. The existence of canonically derived family of surfaces is undoubted. One might refer to [R] for many examples with $P_m = 2$. However it isn't quite clear about the bulk of the set of these families. This paper aims to build some basic facts and to study these families in terms of birational invariants of the total space as well as those of a general fiber. We observed that Kollár proved the following result.

Theorem 0. *(Theorem 6.1 of* [Ko1]*) Let X be a smooth projective 3-fold of general type. If $m \geq 3$ and $\dim\phi_m(X) \leq 2$, then $q(X) := \dim H^1(X, \mathcal{O}_X) \leq 3$.*

As was pointed out by Kollár, one can get $q(X) = 0$ whenever $m \gg 0$ under the condition of Theorem 0. Unfortunately, the bound of m would be much bigger by virtue of his method. Since, in our case, $\dim\phi_m(X) = 1$, we should be able to get more explicit information even if m is small. On the basis of a detailed classification (Theorems 3.3, 3.4), we obatained the following results in this paper.

Theorem 1. *Let $f : X \longrightarrow C$ be a derived family of surfaces from the m-canonical pencil $|mK_X|$ of a smooth projective 3-fold X of general type. Suppose $m \geq 3$. Then C is either an elliptic curve or $\mathbb{P}^1$ and f has the following properties.*

(i) Either $q(X) \leq 1$ or $p_g(X) \leq 1$.

(ii) $q(X) \leq 2$ whenever $m \geq 11$.

(iii) $q(X) \leq 2$ whenever $m \geq 7$ and $p_g(X) > 0$.

(iv) $p_g(X) \leq 1$ whenever $m \geq 7$. $p_g(X) \leq 2$ whenever $5 \leq m \leq 6$. $p_g(X) \leq 3$ whenever $m = 4$. $p_g(X) \leq 5$ whenever $m = 3$.

(v) If $q(X) = 3$, then either $P_m(X) = 2$ or $\dim\phi_{m+1}(X) \geq 2$.

Theorem 2. *Let $f : X \longrightarrow C$ be a derived family of surfaces from the bicanonical pencil $|2K_X|$ of a smooth projective 3-fold X of general type. Then C is either an elliptic curve or $\mathbb{P}^1$ and f has the following properties.*

(i) Either $q(X) \leq 2$ or $p_g(X) \leq 2$.

(ii) If $q(X) \geq 3$, then either $P_2(X) = 2$ or $\dim\phi_3(X) \geq 2$.

In the final section, we would like to give an appendix to Kollár's method on how to determine the bounds of m so as to get $q(X) \leq 1$. The bounds obtained are much better than that of Kollár. However, we feel that they are still far from being the optimal ones. The result is as follows.

Corollary 3. *Let $f : X \longrightarrow C$ be a derived family of surfaces from the m-canonical pencil $|mK_X|$ of a smooth projective 3-fold X of general type. Then*

(i) $q(X) \leq 1$ whenever $m \geq 82$.

(ii) $q(X) = 0$ whenever $m \geq 143$.

1. Preliminaries

1.1 Convention. Let X be a normal projective variety of dimension d. We denote by $\mathrm{Div}(X)$ the group of Weil divisors on X. An element $D \in \mathrm{Div}(X) \otimes \mathbb{Q}$ is called a $\mathbb{Q}$-*divisor*. A $\mathbb{Q}$-divisor D is said to be $\mathbb{Q}$-*Cartier* if mD is a Cartier divisor for some positive integer m. For a $\mathbb{Q}$-Cartier divisor D and an irreducible curve $C \subset X$, we can define the intersection number $D \cdot C$ in a natural way. A $\mathbb{Q}$-Cartier divisor D is called *nef* (namely *numerically effective*) if $D \cdot C \geq 0$ for any effective curve $C \subset X$. A nef divisor D is called *big* if $D^d > 0$. We say that X is $\mathbb{Q}$-*factorial* if every Weil divisor on X is $\mathbb{Q}$-Cartier. For a Weil divisor D on X, write $\mathcal{O}_X(D)$ as the corresponding reflexive sheaf. Denote by K_X a canonical divisor of X, which is a Weil divisor. X is called *minimal* if K_X is a nef $\mathbb{Q}$-Cartier divisor. X is said to be of general type if $\mathrm{kod}(X) = \dim(X)$. For a positive integer m, we set $\omega_X^{[m]} := \mathcal{O}_X(mK_X)$. We use [R] as a nice reference for the definition of *canonical, terminal singularities.* According to both [KMM] and [K-M], any given smooth projective 3-fold Y of general type has a minimal model X which has only $\mathbb{Q}$-factorial terminal singularities.

1.2 Vanishing theorems. Let $D = \sum a_i D_i$ be a $\mathbb{Q}$-divisor on X where the $D_i's$ are distinct prime divisors and $a_i \in \mathbb{Q}$. We define

the round-down $\llcorner D \lrcorner := \sum \llcorner a_i \lrcorner D_i$, where $\llcorner a_i \lrcorner$ is the integral part of a_i.

the round-up $\ulcorner D \urcorner := -\llcorner -D \lrcorner$.

the fractional part $\{D\} := \ulcorner D - \llcorner D \lrcorner \urcorner$.

Throughout this paper, we will use the Kawamata-Viehweg vanishing theorem ([Ka1], [KMM] and [V]) in the following forms.

Theorem 1.1. *Let X be a smooth complete variety, $D \in \mathrm{Div}(X) \otimes \mathbb{Q}$. Assume the following two conditions:*

(i) D is nef and big;

(ii) the fractional part of D has supports with only normal crossings.

Then $H^i(X, \mathcal{O}_X(K_X + \ulcorner D \urcorner)) = 0$ for all $i > 0$.

Theorem 1.2. *Let X be a normal projective variety with only canonical singularities. Let D be a $\mathbb{Q}$-Cartier Weil divisor such that D is nef and big. Then $H^i(X, \mathcal{O}_X(K_X + D)) = 0$ for all $i > 0$.*

1.3 Semi-positivity. Let C be a smooth projective curve and $\mathcal{E}$ be a vector bundle on C. We call

$$\mu(\mathcal{E}) := \frac{\deg(\mathcal{E})}{\mathrm{rk}(\mathcal{E})}$$

the slope of $\mathcal{E}$. According to [H-N], there is the Harder-Narasimhan filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{n-1} \subset \mathcal{E}_n = \mathcal{E},$$

where the quotient $\mathcal{E}_i/\mathcal{E}_{i-1}$ is a semistable vector bundle and

$$\mu(\mathcal{E}_i/\mathcal{E}_{i-1}) > \mu(\mathcal{E}_{i+1}/\mathcal{E}_i)$$

for all i. We define

$$\mu_{min}(\mathcal{E}) := \mu(\mathcal{E}/\mathcal{E}_{n-1}),$$

which is called *the minimal slope of* $\mathcal{E}$.

Definition 1.3. The vector bundle $\mathcal{E}$ is said to be *semi-positive* if $\mu_{min}(\mathcal{E}) \geq 0$.

According to [Ka2], [Ko2], [N] and [O], we have the following

Fact 1.4. *Let X be a smooth projective 3-fold and $f : X \longrightarrow C$ be a proper morphism with connected fibers onto a smooth projective curve C. Then both $f_*\omega_{X/C}^{\otimes m}$ and $R^i f_*\omega_{X/C}$ are semi-positive vector bundles on C for all $m > 0$ and $i > 0$. In particular,*

$$\deg f_*\omega_{X/C}^{\otimes m} \geq 0 \text{ and } \deg R^i f_*\omega_{X/C} \geq 0.$$

1.4 Basic formulae. Let X be a smooth projective 3-fold and $f : X \longrightarrow C$ be a fibration onto the smooth projective curve C. Denote $b := g(C)$. From the spectral sequence

$$E_2^{p,q} := H^p(C, R^q f_*\omega_X) \Longrightarrow E^n := H^n(X, \omega_X),$$

one obtains the following formulae

$$q(X) := h^1(X, \mathcal{O}_X) = b + h^1(C, R^1 f_*\omega_X) \tag{1.1}$$

$$h^2(\mathcal{O}_X) = h^1(C, f_*\omega_X) + h^0(C, R^1 f_*\omega_X). \tag{1.2}$$

2. Lemmas

Lemma 2.1. *Let X be a smooth projective 3-fold of general type. $m \geq 2$ is an integer. Suppose that $|mK_X|$ is composed of a pencil of surfaces. Keep the same notations as in the first page of this paper. We have a derived fibration $f_m : X' \longrightarrow C$. Then C is either an elliptic curve or $\mathbb{P}^1$.*

Proof. Suppose $b > 0$. Then ϕ_m is a morphism. We have a derived fibration

$$f := f_m : X \longrightarrow C.$$

Let $\mathcal{E}_0$ be the saturated sub-bundle of $f_*\omega_X^{\otimes m}$ which is generated by $H^0(C, f_*\omega_X^{\otimes m})$. Since $|mK_X|$ is composed of a pencil and ϕ_m factors through f, $\mathcal{E}_0$ should be a line bundle on C. Denote $\mathcal{E} := f_*\omega_X^{\otimes m}$. Then we have the following extension

$$0 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_1 \longrightarrow 0$$

and the exact sequence

$$f_*\omega_{X/C}^{\otimes m} \longrightarrow \mathcal{E}_1 \otimes \omega_C^{\otimes -m} \longrightarrow 0.$$

Note that $r := \mathrm{rk}(\mathcal{E}) = h^0(F, mK_F) \geq 2$ because the general fiber F is a smooth projective surface of general type. According to Fact 1.4, $f_*\omega_{X/C}^{\otimes m}$ is semi-positive. Therefore $\deg(\mathcal{E}_1 \otimes \omega_C^{\otimes -m}) \geq 0$, i.e.

$$\deg \mathcal{E}_1 \geq 2m(r-1)(b-1).$$

We have

$$\begin{aligned} h^1(\mathcal{E}_0) &\geq h^0(\mathcal{E}_1) \geq \deg \mathcal{E}_1 + (r-1)(1-b) \\ &\geq (2m-1)(r-1)(b-1). \end{aligned} \tag{2.1}$$

Suppose $h^1(\mathcal{E}_0) > 0$. Since $\deg \mathcal{E}_0 > 0$, according to the Clifford's theorem, we have

$$\deg \mathcal{E}_0 \geq 2h^0(\mathcal{E}_0) - 2 \geq h^0(\mathcal{E}_0) = P_m(X) \geq 2.$$

On the other hand, we have

$$h^1(\mathcal{E}_0) = h^0(\mathcal{E}_0) - \deg(\mathcal{E}_0) + b - 1 \leq b - 1. \tag{2.2}$$

Thus, by (2.1) and (2.2), we get

$$b - 1 \geq (2m-1)(r-1)(b-1).$$

The only possibility is $b = 1$. When $h^1(\mathcal{E}_0) = 0$, we also automatically have $b = 1$ from (2.1). The proof is complete. $\square$

Lemma 2.2. *Let $\mathcal{E}$ be a vector bundle of rank r on a smooth projective curve C. Suppose $\mathcal{E} \otimes \omega_C^{-1}$ is semi-positive. Then we have $h^1(C, \mathcal{E}) \leq r$.*

Proof. Suppose there are $r+1$ independant sections

$$s_1, s_2, \cdots, s_{r+1} \in H^0(C, (\mathcal{E} \otimes \omega_C^{-1})^\vee) \cong H^1(C, \mathcal{E}).$$

Denote $\mathcal{E}' := (\mathcal{E} \otimes \omega_C^{-1})^\vee$. For any point $x \in C$, the stalk $\mathcal{E}'_x$ is an $\mathcal{O}_{C,x}$-module of rank r. This means that $s_{1,x}$, $s_{2,x}$, $\cdots$, $s_{r+1,x}$ are algebraically dependant in $\mathcal{E}'_x$. Thus there are $r+1$ nontrivial germs

$$f_1, f_2, \cdots, f_{r+1} \in \mathcal{O}_{C,x}$$

such that $\sum_{i=1}^{r+1} f_i(x) s_{i,x}(x) = 0$. Now set

$$s := \sum_{i=1}^{r+1} f_i(x) s_i \in H^0(C, (\mathcal{E} \otimes \omega_C^{-1})^\vee).$$

s is a non-zero section. Otherwise s_1, $\cdots$, s_{r+1} are dependant. Because s vanishes at x, s defines a line bundle $\mathcal{L}$ which has positive degree. So $\mathcal{E} \otimes \omega_C^{-1}$ has a quotient bundle with negative degree. This contradicts to the semi-positivity of $\mathcal{E} \otimes \omega_C^{-1}$. The proof is completed. $\square$

Corollary 2.3. *Let $f : X \longrightarrow C$ be a fibration from a smooth projective 3-fold X onto a smooth projective curve C. Let F be a general fibre of f and set $b := g(C)$. Then $q(X) \leq b + q(F)$.*

Proof. This is a direct result from Fact 1.4, (1.1) and Lemma 2.2.

3. Proof of the main theorems

Since the behavior of pluricanonical maps is birationally invariant, we may suppose that X is a normal projective minimal 3-fold of general type with only $\mathbb{Q}$-factorial terminal singularities. We make this assumption so as to utilize vanishing theorems. Now suppose that $|mK_X|$ is composed of a pencil of surfaces. We can use the same set up as in the first page of this paper. An extra point is that we can take the modification $\pi_m : X' \longrightarrow X$ such that $\pi_m^*(mK_X)$ has supports with only normal crossings. We keep the same notations. Then we get a derived fibration $f_m : X' \longrightarrow C$. Denote by F a general fiber of f_m and by b the genus of C. Sometimes we simply denote f_m by f and π_m by π respectively.

Proposition 3.1. *Let X be a normal projective minimal 3-fold of general type with only $\mathbb{Q}$-factorial terminal singularities. Suppose $|mK_X|$ is composed of a pencil of surfaces. If $p_g(X) > 0$, then $p_g(F) = 1$ under one of the following conditions.*

(1) $m = 2$, $b = 0$ and $P_m(X) \geq 4$.
(2) $m \geq 3$, $b = 0$ and $P_m(X) \geq 3$.
(3) $m = 2$, $b = 1$ and $P_m(X) \geq 3$.
(4) $m \geq 3$, $b = 1$ and $P_m(X) \geq 2$.

Proof. Because $p_g(X') = p_g(X) > 0$, we can choose an effective divisor $D_1 \in |K_{X'}|$. We write

$$K_{X'} = \pi^*(K_X) + \sum a_i E_i,$$

where $a_i \in \mathbb{Q}^+$, E_i is an exceptional prime divisor for all i. We note that

$$\pi^*(K_X) = K_{X'} - \sum a_i E_i = D_1 - \sum a_i E_i$$

is an effective $\mathbb{Q}$-divisor. So we have

$$\pi^*(K_X) = \sum b_i G_i = G_v + G_h,$$

where $b_i \in \mathbb{Q}^+$, G_i is a prime divisor on X' for all i, the support of G_v is contained in fibers of f and G_h is the horizontal part of $\pi^*(K_X)$. Both G_v and G_h are effective $\mathbb{Q}$-divisors. So we have

$$(m-1)\pi^*(K_X) = (m-1)G_v + (m-1)G_h$$

$$m\pi^*(K_X) = mG_v + mG_h.$$

Suppose M_m is the movable part of $|mK_{X'}|$. Then $M_m \leq_{\mathbb{Q}} mG_v$ because the support of M_m is vertical. Thus we can write

$$mG_v = M_m + G'_v = M_m + \sum c_i G'_i,$$

where $c_i \in \mathbb{Q}^+$, G'_i is a prime divisor on X' for all i. Therefore

$$m\pi^*(K_X) = M_m + (G'_v + mG_h),$$

where $G'_v + mG_h$ is an effective $\mathbb{Q}$-divisor. We can suppose

$$M_m \sim_{\text{lin}} \sum_{i=1}^{a} F_i \sim_{\text{num}} aF,$$

where

$$a = \begin{cases} P_m(X) - 1, & \text{if } b = 0, \\ P_m(X), & \text{if } b = 1. \end{cases}$$

Thus we have

$$\pi^*(K_X) \sim_{\text{num}} \frac{a}{m}F + \frac{1}{m}G'_v + G_h$$

$$(m-1)\pi^*(K_X) \sim_{\text{num}} \frac{a(m-1)}{m}F + \frac{m-1}{m}G'_v + (m-1)G_h,$$

Denote $a' := \frac{a(m-1)}{m}$. We can see that $a' > 1$ under one of the assumptions within (1) through (4) of the proposition. So

$$(m-1)\pi^*(K_X) - F - \frac{m-1}{ma'}G'_v - \frac{m-1}{a'}G_h \sim_{\text{num}} (m-1)(1-\frac{1}{a'})\pi^*(K_X)$$

is nef and big and its fractional part has supports with only normal crossings. According to the Kawamata-Viehweg vanishing theorem, we get

$$H^1(X', K_{X'} + \ulcorner(m-1)\pi^*(K_X) - \frac{m-1}{ma'}G'_v - \frac{m-1}{a'}G_h\urcorner - F) = 0.$$

Set $G'' := \ulcorner(m-1)\pi^*(K_X) - \frac{m-1}{ma'}G'_v - \frac{m-1}{a'}G_h\urcorner$. Then $G'' \leq \ulcorner(m-1)\pi^*(K_X)\urcorner$ and so

$$K_{X'} + G'' \leq K_{X'} + \ulcorner(m-1)\pi^*(K_X)\urcorner.$$

Therefore we have

$$\dim\Phi_{|K_{X'}+G''|}(F) = 0$$

for a general fiber F. From the exact sequence

$$0 \longrightarrow \mathcal{O}_{X'}(K_{X'} + G'' - F) \longrightarrow \mathcal{O}_{X'}(K_{X'} + G'') \longrightarrow \mathcal{O}_F(K_F + G''|_F) \longrightarrow 0,$$

we get the surjective map

$$H^0(X', K_{X'} + G'') \longrightarrow H^0(F, K_F + G''|_F).$$

This means

$$|K_{X'} + G''|\,\big|_F = \big|\ K_F + G''|_F\ \big|\,.$$

Noting that

$$G''|_F = \ulcorner(m-1)\pi^*(K_X) - \frac{m-1}{ma'}G'_v - \frac{m-1}{a'}G_h\urcorner|_F = \ulcorner(m-1)(1-\frac{1}{a'})G_h\urcorner|_F$$

is an effective divisor, we have $h^0(F, K_F + G''|_F) \geq 2$ whenever $p_g(F) \geq 2$. This would lead to $\dim\phi_m(F) \geq 1$, which is impossible. So we should have $p_g(F) = 1$ because $p_g(F) > 0$ under the assumption $p_g(X) > 0$. $\square$

Remark 3.2. The assumption $p_g(X) > 0$ in Proposition 3.1 is important. If $p_g(X) = 0$, the above method is invalid because we don't know whether $G''|_F$ is effective.

Theorem 3.3. *Let $f : X \longrightarrow C$ be a derived family of surfaces from the m-canonical pencil $|mK_X|$ of a smooth projective 3-fold X of general type. Let F be a general fiber of f and denote $b := g(C)$. Suppose $m \geq 3$. Then one of the following occurs:*

(A1) $m = 5,\ 6,\ p_g(X) = 2,\ q(X) = 0,\ b = 0$ *and* $p_g(F) = q(F) = 1$.
(A2) $m = 4,\ 2 \leq p_g(X) \leq 3,\ q(X) = 0,\ b = 0$ *and* $p_g(F) = 1$.
(A3) $m = 3,\ 2 \leq p_g(X) \leq 5,\ q(X) \leq 1,\ b = 0$ *and* $p_g(F) = 1$.
(B0) $p_g(X) = 1,\ q(X) \leq b + 1$ *and* $p_g(F) = 1$.
(B1) $p_g(X) = P_2(X) = \cdots = P_{m-1}(X) = 1,\ P_m(X) = 2,\ \dim\phi_{m+1}(X) \geq 2,$ $b = 0$ *and* $p_g(F) \geq 2$.

(B2) $p_g(X) = P_2(X) = \cdots = P_{m-1}(X) = 1$, $P_m(X) = P_{m+1}(X) = 2$, $\dim\phi_{m+2}(X) \geq 2$, $b = 0$ *and* $p_g(F) \geq 2$.

(B3) $p_g(X) = P_2(X) = \cdots = P_{m-2}(X) = 1$, $P_{m-1}(X) = P_m(X) = 2$, $\dim\phi_{m+1}(X) \geq 2$, $b = 0$ *and* $p_g(F) \geq 2$.

(C0) $p_g(X) = 0$, $q(X) \leq 2$ *and* $q(F) \leq 2$.

(C1) $p_g(X) = 0$, $\dim\phi_{m+1}(X) \geq 2$ *and* $q(F) \geq 2$.

(C2) $p_g(X) = 0$, $P_m(X) = 2$, $\dim\phi_{m+2}(X) \geq 2$, $b = 1$ *and* $q(F) = 2$.

(C3) $p_g(X) = 0$, $P_m(X) = 2$, $\dim\phi_{2m}(X) \geq 2$, $b = 0$ *and* $q(F) \geq 3$.

(C4) $p_g(X) = 0$, $P_m(X) = 2$, $P_{2m}(X) = 3$, $\dim\phi_{2m+1}(X) = 3$, $b = 0$ *and* $q(F) \geq 3$.

Proof. We formulate the proof through three steps. Though the proof is slightly longer, it's a case by case discussion.

Step 1. $p_g(X) \geq 2$

Suppose $b = 1$. In this situation, we see that the movable part of $|3K_X|$ defines a morphism. Because $p_g(X) \geq 2$, $\dim\phi_1(X) = 1$ and both ϕ_1 and ϕ_3 derive the same fibration $f : X \longrightarrow C$. So the movable part of $|K_X|$ is also base point free. Let M_1 be the movable part of $|K_X|$. Then $M_1 \sim_{\text{lin}} \sum F_i$. Let F be a general fiber of f. Because the singularities on X are all isolated, F is a smooth projective surface of general type. By Theorem 2.2, we see that $H^1(X, 2K_X) = 0$. Therefore we have

$$|2K_X + \sum F_i||_F = |2K_F|.$$

Because $p_g(F) > 0$, $\Phi_{|2K_F|}$ is generically finite by Theorem 1 of [X1]. This means that ϕ_3 is generically finite and so is ϕ_m for $m \geq 4$. So we only have to consider the case when $b = 0$.

Now we have a fibration $f : X' \longrightarrow \mathbb{P}^1$. Because $p_g(X) \geq 2$, we have $P_3(X) \geq 4$. Thus, by Proposition 3.1, we see that $p_g(F) = 1$. In this situation, Kollár's technique (the proof of Corollary 4.8 in [Ko1]) is still effective. Let $p_g(X) = k + 1$, $k \geq 1$. Since $|K_{X'}|$ is composed of a pencil, we have $\mathcal{O}(k) \hookrightarrow f_*\omega_{X'}$ on $\mathbb{P}^1$. If $k \geq 5$, then

$$\mathcal{O}(5) \hookrightarrow \mathcal{O}(k) \hookrightarrow f_*\omega_{X'}.$$

Thus we have

$$\mathcal{E} := \mathcal{O}(1) \otimes f_*\omega^2_{X'/\mathbb{P}^1} = \mathcal{O}(5) \otimes f_*\omega^2_{X'} \hookrightarrow f_*\omega^3_{X'}.$$

The local sections of $f_*\omega^2_{X'}$ give the bicanonical map of the fiber F and they extend to global sections of $\mathcal{E}$, because $\mathcal{E}$ is generated by global sections. On the other hand, $H^0(\mathbb{P}^1, \mathcal{E})$ can distinguish different fibers of f because $f_*\omega^2_{X'/\mathcal{P}^1}$ is a sum of line bundles with nonnegative degree on $\mathbb{P}^1$. So $H^0(\mathbb{P}^1, \mathcal{E})$ gives a generically finite map on X' and so does $H^0(X', 3K_{X'})$. This contradicts to our assumption of $\dim\phi_3(X) = 1$. Thus we have $k \leq 4$, i.e. $p_g(X) \leq 5$. By virtue of this technique, we have

$$\begin{aligned}
&k = 3,\ 4,\ \mathcal{E} \hookrightarrow f_*\omega^4_{X'} \Longrightarrow m = 3 \Longrightarrow\ (A3)\\
&k = 2,\qquad \mathcal{E} \hookrightarrow f_*\omega^5_{X'} \Longrightarrow m \leq 4 \Longrightarrow\ (A2),\ (A3)\\
&k = 1,\qquad \mathcal{E} \hookrightarrow f_*\omega^7_{X'} \Longrightarrow m \leq 6 \Longrightarrow\ (A1),\ (A2),\ (A3)
\end{aligned}$$

In order to complete the proof for this case, we have to prove $q(X)=0$ for (A1), (A2) and that the only possibility of F in (A1) is $p_g(F)=q(F)=1$. Suppose $q(X)=1$ in cases (A1) and (A2). Then $q(F)=1$ and $R^1f_*\omega_{X'}\cong\omega_{\mathbb{P}^1}$. Because $f_*\omega_{X'}$ is of positive degree, we have $h^1(\mathbb{P}^1,f_*\omega_{X'})=0$. So by (1.2), $h^2(\mathcal{O}_{X'})=0$. Then we have $\chi(\mathcal{O}_{X'})\le -2$. According to Reid's plurigus formula ([R]), we have $P_2(X)\ge 7$. This means $\mathcal{O}(6)\hookrightarrow f_*\omega_{X'}^2$ and $\mathcal{E}\hookrightarrow f_*\omega_{X'}^4$. So ϕ_4 is generically finite, a contradiction. Finally, with regard to (A1), if $q(F)=0$, then $h^2(\mathcal{O}_{X'})=0$. So $\chi(\mathcal{O}_{X'})\le -1$. Thus $P_3(X)\ge 6$ according to Reid. We have $\mathcal{O}(5)\hookrightarrow f_*\omega_{X'}^3$ and $\mathcal{E}\hookrightarrow f_*\omega_{X'}^5$. So ϕ_5 is generically finite, a contradiction.

Step 2. $p_g(X)=1$

When $b=1$ or $b=0$ and $P_m(X)\ge 3$, we have $p_g(F)=1$ according to Proposition 3.1. This leads to (B0). From now on, we can suppose $b=0$, $P_m(X)=2$ and $p_g(F)\ge 2$.

We claim that $P_{m-2}(X)=1$. In fact, if $P_{m-2}(X)>1$, we must have $P_{m-2}(X)=2$. So the movable part of $|(m-2)K_{X'}|$ is a fiber F of f. By Theorem 1.1, we have

$$|K_{X'}+\ulcorner\pi^*(K_X)\urcorner+F|\,\big|_F=|K_F+D|,$$

where $D:=\ulcorner\pi^*(K_X)\urcorner|_F$ is an effective divisor on F. So we see that $\dim\phi_m(X)\ge 2$, a contradiction.

If $P_{m-1}(X)=2$, then we can see from the above argument that $\dim\phi_{m+1}(X)\ge 2$. This leads to (B3).

If $P_{m-1}(X)=1$ and we are not in (B1), then $P_{m+1}(X)=2$ by virtue of Proposition 3.1. We can easily see that $\dim\phi_{m+2}(X)\ge 2$. This leads to (B2).

Step 3. $p_g(X)=0$

We can suppose $b=1$ and $q(F)\ge 2$ or $b=0$ and $q(F)\ge 3$. Otherwise we are in case (C0).

Suppose $b=1$. If $q(F)\ge 3$, we can see that $\dim\phi_{m+1}(X)\ge 2$. In fact, we can write

$$m\pi^*(K_X)\sim_{\mathbb{Q}} aF+E_{\mathbb{Q}}^{(m)},$$

where $a=P_m(X)\ge 2$ and $E_{\mathbb{Q}}^{(m)}$ is an effective $\mathbb{Q}$-divisor. It is obvious that

$$K_{X'}+\ulcorner m\pi^*(K_X)-F-\frac{1}{a}E_{\mathbb{Q}}^{(m)}\urcorner\le (m+1)K_{X'}.$$

Since

$$m\pi^*(K_X)-F-\frac{1}{a}E_{\mathbb{Q}}^{(m)}\sim_{\mathrm{num}} m(1-\frac{1}{a})\pi^*(K_X)$$

is nef and big, we get by the vanishing theorem that

$$H^1(X',K_{X'}+\ulcorner m\pi^*(K_X)-F-\frac{1}{a}E_{\mathbb{Q}}^{(m)}\urcorner)=0.$$

This means that

$$|K_{X'}+\ulcorner m\pi^*(K_X)-\frac{1}{a}E_{\mathbb{Q}}^{(m)}\urcorner|\,\big|_F=\big|\,K_F+\ulcorner m\pi^*(K_X)-\frac{1}{a}E_{\mathbb{Q}}^{(m)}\urcorner|_F\,\big|,$$

where

$$\ulcorner m\pi^*(K_X)-\frac{1}{a}E_{\mathbb{Q}}^{(m)}\urcorner|_F=\ulcorner(1-\frac{1}{a})E_{\mathbb{Q}}^{(m)}\urcorner|_F$$

is an effective divisor. According to [X2], $|K_F|$ gives a generically finite map. So we see that $\dim\phi_{m+1}(F) = 2$ and thus $\dim\phi_{m+1}(X) \geq 2$. This leads to (C1). If $q(F) = 2$ and $P_m(X) \geq 3$, we can still see that $\dim\phi_{m+1}(X) \geq 2$. In this situation, $a \geq 3$. Let F_1 and F_2 be two distinct general fibers of f. Then we see that

$$m\pi^*(K_X) - F_1 - F_2 - \frac{2}{a}E_{\mathbb{Q}}^{(m)} \sim_{\text{num}} m(1 - \frac{2}{a})\pi^*(K_X)$$

is nef and big. So we have the following surjective map

$$\begin{aligned} &H^0(X', K_{X'} + \ulcorner m\pi^*(K_X) - \frac{2}{a}E_{\mathbb{Q}}^{(m)}\urcorner) \longrightarrow \\ &H^0(F_1, K_{F_1} + D_1) \oplus H^0(F_2, K_{F_2} + D_2) \longrightarrow 0, \end{aligned}$$

where

$$D_i = \ulcorner m\pi^*(K_X) - \frac{2}{a}E_{\mathbb{Q}}^{(m)}\urcorner|_{F_i} = \ulcorner m(1 - \frac{2}{a})E_{\mathbb{Q}}^{(m)}\urcorner|_{F_i}$$

is effective for all i. This means that

$$|K_{X'} + \ulcorner m\pi^*(K_X) - \frac{2}{a}E_{\mathbb{Q}}^{(m)}\urcorner|$$

can distinguish two different fibers of f and $\dim\phi_{m+1}(F_i) \geq 1$. We again see that $\dim\phi_{m+1}(X) \geq 2$. This leads to (C1). If $q(F) = 2$ and $P_m(X) = 2$, we can use a parallel argument to that in the proof of the case $b = 1$ of Step 1 to see that $\dim\phi_{m+2}(X) \geq 2$. This corresponds to (C2).

Suppose $b = 0$ and $q(F) \geq 3$. If $P_m(X) \geq 3$, we can use the same argument as in the case $b = 1$ of Step 3 to see that $\dim\phi_{m+1}(X) \geq 2$. This leads to (C1). What remains to be studied is the case $P_m(X) = 2$. This is the most frustrating case. Anyway, it is easy to see that $\dim\phi_{2m+1}(X) \geq 2$ in this case. Actually, one only has to consider the system

$$|K_{X'} + \ulcorner m\pi^*(K_X)\urcorner + F|.$$

We can see that

$$|K_{X'} + \ulcorner m\pi^*(K_X)\urcorner + F|\ \big|_F = \big|\ K_F + \ulcorner m\pi^*(K_X)\urcorner|_F\ \big|,$$

where $\ulcorner m\pi^*(K_X)\urcorner|_F$ is effective. This means

$$\dim\phi_{2m+1}(X) \geq \dim\phi_{2m+1}(F) = 2.$$

Now if $\dim\phi_{2m}(X) \geq 2$, we are in (C3). If $\dim\phi_{2m}(X) = 1$, we have the following claim which shows that we are in either (C1) or (C4). We note that $P_{2m}(X) \geq 3$.

Claim. If $b = 0$, $q(F) \geq 3$, $P_m(X) = 2$, $P_{2m}(X) \geq 4$ and $\dim\phi_{2m}(X) = 1$. Then $\dim\phi_{m+1}(X) \geq 2$. This leads to (C1).

Since $\dim\phi_{2m}(X) = 1$, we can see that both ϕ_{2m} and ϕ_m derive the same fibration $f : X' \longrightarrow \mathbb{P}^1$. We can write

where D_v is a vertical $\mathbb{Q}$-divisor with respect to the fibration f and D_h is the horizontal part. The supports of D_v and D_h are contained in the fixed part of $|mK_{X'}|$. D_v and D_h are both effective $\mathbb{Q}$-divisors. Similarly, we can write

$$2m\pi^*(K_X) \sim_{\mathbb{Q}} \sum_{i=1}^{a_2} F_i + (D'_v + D'_h),$$

where $a_2 \geq P_{2m}(X) - 1 \geq 3$, D'_v is a vertical effective $\mathbb{Q}$-divisor and D'_h is a horizontal effective $\mathbb{Q}$-divisor. Since the support of D'_h is contained in the fixed part of $|2mK_{X'}|$, we can see that $D'_h = 2D_h$. Now we have

$$2m\pi^*(K_X) \sim_{\text{num}} a_2F + D'_v + 2D_h,$$

$$m\pi^*(K_X) \sim_{\text{num}} \frac{a_2}{2}F + \frac{1}{2}D'_v + D_h.$$

So

$$m\pi^*(K_X) - F - \frac{1}{a_2}D'_v - \frac{2}{a_2}D_h \sim_{\text{num}} m(1 - \frac{2}{a_2})\pi^*(K_X)$$

is nef and big. This means, according to the vanishing theorem, that

$$H^1(X', K_{X'} + \ulcorner m\pi^*(K_X) - F - \frac{1}{a_2}D'_v - \frac{2}{a_2}D_h\urcorner) = 0.$$

Denote $M := \ulcorner m\pi^*(K_X) - \frac{1}{a_2}D'_v - \frac{2}{a_2}D_h\urcorner$. Then $K_{X'} + M \leq (m+1)K_{X'}$. Then

$$|K_{X'} + M|\ \Big|_F = \Big|\ K_F + M|_F\ \Big|,$$

where $M|_F = \ulcorner(1 - \frac{2}{a_2})D_h\urcorner|_F$ is an effective divisor on F. So

$$\dim\phi_{m+1}(X) \geq \dim\Phi_{|K_{X'}+M|}(F) = 2.$$

The proof is complete. $\square$

Theorem 3.4. *Let $f : X \longrightarrow C$ be a derived family of surfaces from the bicanonical pencil $|2K_X|$ of a smooth projective 3-fold X of general type. Let F be a general fiber of f and denote $b := g(C)$. Then one of the following occurs.*

(A0)' $p_g(X) > 0$, $q(X) \leq b + 1$ *and* $p_g(F) = 1$.
(A1)' $1 \leq p_g(X) \leq 2$, $2 \leq P_2(X) \leq 3$, $\dim\phi_3(X) \geq 2$ *and* $p_g(F) \geq 2$.
(A2)' $p_g(X) = 1$, $P_2(X) = P_3(X) = 2$, $\dim\phi_4(X) \geq 2$, $b = 0$ *and* $p_g(F) \geq 2$.
(B0)' $p_g(X) = 0$, $q(X) \leq 2$ *and* $q(F) \leq 2$.
(B1)' $p_g(X) = 0$, $\dim\phi_3(X) \geq 2$ *and* $q(F) \geq 2$.
(B2)' $p_g(X) = 0$, $P_2(X) = 2$, $\dim\phi_4(X) \geq 2$, $b = 1$ *and* $q(F) = 2$.
(B3)' $p_g(X) = 0$, $P_2(X) = 2$, $\dim\phi_4(X) \geq 2$, $b = 0$ *and* $q(F) \geq 3$.
(B4)' $p_g(X) = 0$, $P_2(X) = 2$, $P_4(X) = 3$, $\dim\phi_5(X) \geq 2$, $b = 0$ *and* $q(F) \geq 3$.

Proof. The proof is parallel to that of Theorem 3.3 except that we have more cases here. In order to avoid unnecessary redundancy, we only give the proof where it is different from the respective part in the proof of Theorem 3.3.

Step 1. $p_g(X) \geq 2$.

In this case, we always have $P_2(X) \geq 3$. When $b = 0$ and $P_2(X) \geq 4$ or $b = 1$, we see from Propositioin 3.1 that $p_g(F) = 1$. This leads to (A0)'.

So we only have to consider the case with $b = 0$, $P_2(X) = 3$ and $p_g(F) \geq 2$. In this situation, we see that the movable part of $|2K_{X'}|$ contains exactly 2 fibers of f. Since $P_3(X) \geq 4$, Proposition 3.1 gives $\dim\phi_3(X) \geq 2$. This corresponds to (A1)'.

Step 2. $p_g(X) = 1$.

Excluding the situation (A0)' while observing Proposition 3.1, we only have to consider the case with $p_g(F) \geq 2$ and with the following extra properties:

$$b = 0,\ P_2(X) \leq 3 \text{ or } b = 1,\ P_2(X) = 2.$$

When $b = 0$ and $P_2(X) = 3$ or $b = 1$ and $P_2(X) = 2$, we know from Proposition 3.1 that $\dim\phi_3(X) \geq 2$. This leads to (A1)'.

When $b = 0$, $P_2(X) = 2$, $p_g(F) \geq 2$ and $P_3(X) \geq 3$, we see from Proposition 3.1 that $\dim\phi_3(X) \geq 2$. This also corresponds to (A1)'. Otherwise we always have $\dim\phi_4(X) \geq 2$ because $P_4(X) \geq 3$. This is just (A2)'.

Step 3. $p_g(X) = 0$.

The argument in the proof of Theorem 3.3 is still effective in this case. We can see that (C0) through (C4) correspond to (B0)' through (B4)', respectively. We omit the proof. $\square$

Now we can see that Theorem 1 (i), (iv) and (v), Theorem 2 are direct results from Theorem 3.3 and Theorem 3.4. In order to complete the proof of Theorem 1, we only have to show $q(X) \leq 2$ whenever $m \geq 11$ or $m \geq 7$ and $p_g(X) > 0$.

Proposition 3.5. *Let X be a minimal projective 3-fold of general type with only $\mathbb{Q}$-factorial terminal singularities. Suppose $q(X) \geq 3$, $P_{k_0}(X) > 0$ and $P_{k_2}(X) \geq 2$. Then* $\dim\phi_{k_0+k_2+1}(X) \geq 2$.

Proof. Choose a 1-dimensional subsystem $\Lambda \subset |k_2K_X|$ while taking a birational modification $\pi : X' \longrightarrow X$ such that the pencil Λ defines a morphism $g : X' \longrightarrow \mathbb{P}^1$. We can even take further modification to π so that $\pi^*(k_2K_X)$ has supports with only normal crossings. Taking the Stein factorization of g, then we get a derived fibration $p : X' \longrightarrow C_1$. We note that this fibration is different from the one which was defined at the first page of this paper. Denote $b_1 := g(C_1)$. Let M be the movable part of the pencil. We obviously have $M \leq k_2K_{X'}$. We can write $M \sim_{\text{lin}} \sum_{i=1}^{a_1} F_i$, where $a_1 \geq 1$ and F_i is a fiber of p for all i. We also note that $a_1 = 1$ if and only if $b_1 = 0$. A general fiber F is a smooth projective surface of general type.

Suppose $b_1 > 0$. Then $|M|$ is base point free on X. Because X has only isolated singularities, F is smooth. We study the system $|tK_X + M|$ where $t \geq 2$. We know that M contains at least two components F_1 and F_2. By Theorem 1.2, we see that

$$H^0(X, tK_X + M) \longrightarrow H^0(F_1, tK_{F_1}) \oplus H^0(F_2, tK_{F_2})$$

is surjective. This means that $\Phi_{|tK_X+M|}$ can distinguish F_1 and F_2 and the restriction to F_i is at least a bicanonical map. We know that $\dim\Phi_{|tK_{F_i}|}(F_i) \geq 1$ for all $t \geq 2$. Noting that the image of X through $\Phi_{|tK_X+M|}$ is irreducible, we see that $\dim\Phi_{|tK_X+M|}(X) \geq 2$. So $\dim\phi_{t+k_2}(X) \geq 2$. Thus $\dim\phi_{k_0+k_2+1}(X) \geq 2$.

Suppose $b = 0$. By Corollary 2.3, we have $q(F) \geq 3$. In this case, $M \sim_{\text{lin}} F$. We have

where $\ulcorner k_0\pi^*(K_X)\urcorner|_F$ is effective. Thus $\dim\phi_{k_0+k_2+1}(X) \geq \dim\phi_{k_0+k_2+1}(F) = 2$. $\square$

Lemma 3.6. *Let X be a smooth projective 3-fold of general type. If $q(X) \geq 3$, then either*

$$P_2(X) > 0 \text{ and } P_4(X) \geq 2$$

or

$$P_k(X) \geq 2 \text{ for all } k \geq 5.$$

Proof. This is a byproduct from the proof of both Theorem 6.1, [Ko1] and Proposition 4.3, [Ko1]. $\square$

Proposition 3.7. *Let X be a minimal projective 3-fold of general type with only $\mathbb{Q}$-factorial terminal singularities. Suppose $q(X) \geq 3$, $P_2(X) > 0$ and $P_4(X) \geq 2$. Then*

(1) $\dim\phi_7(X) \geq 2$. $\dim\phi_m(X) \geq 2$ *for all* $m \geq 9$.

(2) If $\dim\phi_8(X) = 1$, *then* $p_g(X) = 0$, $P_2(X) = 1$, $P_4(X) = 2$ *and* $P_8(X) = 3$.

Proof. (1) Let $k_0 = 2$ and $k_2 = 4$. Proposition 3.5 gives $\dim\phi_7(X) \geq 2$. So $\dim\phi_{2l+7}(X) \geq 2$ for all $l \in \mathbb{Z}^+$. Let $k_0 = 2$ and $k_2 = 7$. Applying Proposition 3.5 again, we get $\dim\phi_{10}(X) \geq 2$. Thus $\dim_{2l+10}(X) \geq 2$ for all $l \in \mathbb{Z}^+$.

(2) We study ϕ_8. We have $P_8(X) \geq 3$. If $P_8(X) \geq 4$ and $\dim\phi_8(X) = 1$, we want to deduce a contradiction. We know that both ϕ_4 and ϕ_8 derive the same fibration $f : X' \longrightarrow C$ which was described in the first page of this paper. If $b > 0$, it is easy to see that $\dim\phi_8(X) \geq 2$ by a standard argument which has been used many times in this paper. So we can suppose $b = 0$. Since $q(X) \geq 3$, we have $q(F) \geq 3$. Suppose M_4, M_8 are the movable parts of $|4K_{X'}|$, $|8K_{X'}|$ respectively. Then we have

$$4\pi^*(K_X) \sim_{\mathbb{Q}} M_4 + E_4,$$

$$8\pi^*(K_X) \sim_{\mathbb{Q}} M_8 + E_8,$$

where E_4 and E_8 are effective $\mathbb{Q}$-divisors. Let E_v, E_v' be the vertical parts of E_4, E_8 and E_h, E_h' be the horizontal parts of E_4, E_8 respectively. Because the support of E_h is contained in the fixed part of $|4K_{X'}|$ and the support of E_h' is contained in the fixed part of $|8K_{X'}|$, we see that $E_h' = 2E_h$. Now we have

$$8\pi^*(K_X) \sim_{\text{num}} a_8F + E_v' + 2E_h,$$

where $a_8 \geq 3$. It follows that

$$4\pi^*(K_X) \sim_{\text{num}} \frac{a_8}{2}F + \frac{1}{2}E_v' + E_h.$$

Thus

$$4\pi^*(K_X) - F - \frac{1}{a_8}E_v' - \frac{2}{a_8}E_h \sim_{\text{num}} 4(1 - \frac{2}{a_8})\pi^*(K_X)$$

is a nef and big $\mathbb{Q}$-divisor. Denote

$$G := \ulcorner 4\pi^*(K_X) - \frac{1}{a_8}E_v' - \frac{2}{a_8}E_h\urcorner.$$

Then we have $H^1(X', K_{X'}+G-F)=0$. So we see that

$$|K_{X'}+G|\,\big|_F = |K_F + G|_F|,$$

where

$$G|_F = \ulcorner 4\pi^*(K_X) - \frac{1}{a_8}E_v' - \frac{2}{a_8}E_h \urcorner|_F = \ulcorner(1-\frac{2}{a_8})E_h\urcorner|_F$$

is effective. So $\dim\phi_5(X) \geq \dim\Phi_{|K_F+G|_F|}(F)=2$. In particular, $P_5(X)\geq 2$. Now let $k_0=2$ and $k_2=5$. Applying Proposition 3.5, we see that $\dim\phi_8(X)\geq 2$. This contradicts to our assumption. Thus we have seen that $P_8(X)=3$ if $\dim\phi_8(X)=1$. It follows immediately that $P_4(X)=2$ and $P_2(X)=1$. If $P_g(X)>0$, it is very easy to see from Proposition 3.5 that $\dim\phi_8(X)\geq 2$. So we have completed the proof. □

Proposition 3.8. *Let X be a minimal projective 3-fold of general type with only $\mathbb{Q}$-factorial terminal singularities. Suppose $q(X)\geq 3$ and $P_k(X)\geq 2$ for all $k\geq 5$. Then*

(1) $\dim\phi_m(X)\geq 2$ *for all $m\geq 11$.*
(2) If $\dim\phi_{10}(X)=1$, *then $p_g(X)=P_2(X)=P_3(X)=P_4(X)=0$.*
(3) If $\dim\phi_9(X)=1$, *then $p_g(X)=P_2(X)=P_3(X)=0$.*
(4) If $\dim\phi_8(X)=1$, *then $p_g(X)=P_2(X)=0$.*
(5) If $\dim\phi_7(X)=1$, *then $p_g(X)=0$.*

Proof. Let $k_0=5$ and $k_2=t\geq 5$. Proposition 3.5 gives $\dim\phi_{t+6}(X)\geq 2$ for all $t\geq 5$. This leads to (1).

If $p_g(X)>0$, let $k_0=1$ and $k_2=t\geq 5$. Proposition 3.5 gives $\dim\phi_{t+2}(X)\geq 2$ for all $t\geq 5$.

If $P_2(X)>0$, let $k_0=2$ and $k_2=t\geq 5$. Proposition 3.5 gives $\dim\phi_{t+3}(X)\geq 2$ for all $t\geq 5$.

If $P_3(X)>0$, let $k_0=3$ and $k_2=t\geq 5$. Proposition 3.5 gives $\dim\phi_{t+4}(X)\geq 2$ for all $t\geq 5$.

If $P_4(X)>0$, let $k_0=4$ and $k_2=t\geq 5$. Proposition 3.5 gives $\dim\phi_{t+5}(X)\geq 2$ for all $t\geq 5$.

(2), (3), (4) and (5) follow immediately. □

Corollary 3.9. *Let $f: X\longrightarrow C$ be a derived family of surfaces from the m-canonical pencil $|mK_X|$ of a smooth projective 3-fold X of general type. Then*

(1) $q(X)\leq 2$ when $m\geq 11$.
(2) $q(X)\leq 2$ when $m\geq 7$ and $p_g(X)>0$.

Proof. This is obvious from Lemma 3.6, Proposition 3.7 and Proposition 3.8. □

4. Appendix to Kollár's method

Given a smooth projective 3-fold X of general type, it is uncertain whether $\dim\phi_{m+1}(X)\geq\dim\phi_m(X)$ for all $m>0$. Even if $P_m(X)>0$, it is false that $P_{m+1}(X)>0$. This makes it difficult to study some stable property of ϕ_m. That is why Kollár's bound was bigger. Hereby we would like to study in an alternative way. The bounds are better, however still unsatisfactory.

Proposition 4.1. *Let X be a minimal projective 3-fold of general type with only $\mathbb{Q}$-factorial terminal singularities. Suppose $q(X) \geq 2$ and $P_{k_2}(X) \geq 2$. Then* $\dim\phi_m(X) \geq 2$ *for all* $m \geq 4k_2 + 2$.

Proof. Choose a 1-dimensional subsystem $\Lambda \subset |k_2K_X|$ while taking a birational modification $\pi : X' \longrightarrow X$ such that the pencil Λ defines a morphism $g : X' \longrightarrow \mathbb{P}^1$. We can even take further modification to π so that $\pi^*(k_2K_X)$ has supports with only normal crossings. Taking the Stein factorization of g, then we get a derived fibration $p : X' \longrightarrow C_2$. Denote $b_2 := g(C_2)$. Let M be the movable part of the pencil. We obviously have $M \leq k_2K_{X'}$. We can write $M \sim_{\text{lin}} \sum_{i=1}^{a_2} F_i$, where $a_2 \geq 1$ and F_i is a fiber of q. A general fiber F is a smooth projective surface of general type.

If $b_2 > 0$, then we can see that $\dim\phi_{k_2+t}(X) \geq 2$ for all $t \geq 2$ according to the parallel argument in the proof of Proposition 3.5.

If $b_2 = 0$. we study in an alternative way. We have $M \sim_{\text{lin}} F$. Because $q(X) \geq 2$, we have $p_g(F) \geq q(F) \geq 2$. According to Theorem 1.1, we have

$$|K_{X'} + \ulcorner k_2\pi^*(K_X)\urcorner + F| \,\big|_F = \big| \, K_F + \ulcorner k_2\pi^*(K_X)\urcorner|_F \,\big| \, .$$

This means $\dim\phi_{2k_2+1}(F) \geq 1$ because $\ulcorner k_2\pi^*(K_X)\urcorner|_F$ is effective. Suppose M_{2k_2+1} is the movable part of $|(2k_2+1)K_{X'}|$ and M'_{2k_2+1} is the movable part of

$$|K_{X'} + \ulcorner k_2\pi^*(K_X)\urcorner + F|.$$

Then $M'_{2k_2+1} \leq M_{2k_2+1}$. Let M_0 be the movable part of $|K_F|$. Then $h^0(F, M_0) \geq 2$. Considering the following two maps

$$H^0(X', K_{X'} + \ulcorner k_2\pi^*(K_X)\urcorner + F) \xrightarrow{\alpha} H^0(F, K_F + \ulcorner k_2\pi^*(K_X)\urcorner|_F) \longrightarrow 0$$

$$H^0(X', M'_{2k_2+1}) \xrightarrow{\beta} H^0(F, M'_{2k_2+1}|_F),$$

we know that α is surjective and the images of α and β have the same dimension. So

$$\begin{aligned} &h^0(F, M'_{2k_2+1}|_F) \geq \dim_{\mathbb{C}}\text{im}(\beta) = \dim_{\mathbb{C}}\text{im}(\alpha) \\ &= h^0(F, K_F + \ulcorner k_2\pi^*(K_X)\urcorner|_F). \end{aligned}$$

Because

$$M'_{2k_2+1}|_F \leq K_F + \ulcorner k_2\pi^*(K_X)\urcorner|_F,$$

we see that

$$M_0 \leq M'_{2k_2+1}|_F \leq M_{2k_2+1}|_F.$$

For all $t \geq 0$ and two different fibers F_1, F_2, we consider the system

$$|K_{X'} + \ulcorner (t + 2k_2 + 1)\pi^*(K_X)\urcorner + F_1 + F_2|.$$

It is obvious that

$$|K_{X'} + \ulcorner (t + 2k_2 + 1)\pi^*(K_X)\urcorner + F_1 + F_2| \subset |(t + 4k_2 + 2)K_{X'}|$$

From Theorem 1.1, we have the exact sequence

$$\begin{aligned} &H^0(X', K_{X'} + \ulcorner (t+2k_2+1)\pi^*(K_X)\urcorner + F_1 + F_2) \\ &\longrightarrow H^0(F_1, K_{F_1} + G_1) \oplus H^0(F_2, K_{F_2} + G_2) \longrightarrow 0, \end{aligned}$$

where $G_i = (\ulcorner (t+2k_2+1)\pi^*(K_X)\urcorner + F_1 + F_2)|_{F_i}$ for all i. We can see that

$$\begin{aligned} K_{F_i} + G_i &\geq K_{F_i} + \ulcorner t\pi^*(K_X)|_{F_i}\urcorner + M_{2k_2+1}|_{F_i} \\ &\geq K_{F_i} + \ulcorner t\pi^*(K_X)|_{F_i}\urcorner + M_0 \end{aligned}$$

for all i. Furthermore, one can see that

$$\dim\Phi_{|K_{F_i} + \ulcorner t\pi^*(K_X)|_{F_i}\urcorner + M_0|}(F_i) \geq 1.$$

When $t = 0$, it is obvious. When $t > 0$, one need to use the vanishing theorem to prove it. Noting that the image of X' through ϕ_{t+4k_2+2} is irreducible and that

$$\dim\phi_{t+4k_2+2}(F_i) \geq 1$$

for all i, we can see $\dim\phi_{t+4k_2+2}(X) \geq 2$. $\square$

Proposition 4.2. *Let X be a minimal projective 3-fold of general type with only $\mathbb{Q}$-factorial terminal singularities. Suppose $q(X) > 0$ and $P_{k_2}(X) \geq 2$. Then $\dim\phi_m(X) \geq 2$ for all $m \geq 7k_2 + 3$.*

Proof. We keep the same set up as in the proof of Proposition 4.1. We only have to study the case when $b_2 = 0$. We have the fibration $p : X' \longrightarrow \mathbb{P}^1$. We still denote by F a general fiber of p. Since $q(X) > 0$, we get $p_g(F) \geq q(F) \geq 1$. If $p_g(F) \geq 2$, we have seen from the proof of the last proposition that we can get better bounds. The most frustrating case is when $p_g(F) = q(F) = 1$. Let $\sigma : F \longrightarrow F_0$ be the contraction onto the minimal model. According to Theorem 3.1 in [Ci], we know that $|2K_{F_0}|$ is base point free when $p_g(F) > 0$. So the movable part of $|2K_F|$ is just $\sigma^*(2K_{F_0})$. According to Kollár's method, we see that

$$|(5k_2+2)K_{X'}|\ \big|_F \supset |2K_F|.$$

So, if we denote by M_{5k_2+2} the movable part of $|(5k_2+2)K_{X'}|$, we should have

$$M_{5k_2+2}|_F \geq \sigma^*(2K_{F_0}).$$

For all $t \geq 0$ and two different fibers F_1, F_2, we consider the system

$$|K_{X'} + \ulcorner (t+5k_2+2)\pi^*(K_X)\urcorner + F_1 + F_2|.$$

It is obvious that

$$|K_{X'} + \ulcorner (t+5k_2+2)\pi^*(K_X)\urcorner + F_1 + F_2| \subset |(t+7k_2+3)K_{X'}|.$$

From Theorem 1.1, we have the exact sequence

$$\begin{aligned} &H^0(X', K_{X'} + \ulcorner (t+5k_2+2)\pi^*(K_X)\urcorner + F_1 + F_2) \\ &\longrightarrow H^0(F_1, K_{F_1} + G_1') \oplus H^0(F_2, K_{F_2} + G_2') \longrightarrow 0, \end{aligned}$$

where $G_i' = (\ulcorner(t+5k_2+2)\pi^*(K_X)\urcorner + F_1 + F_2)|_{F_i}$ for all i. We can see that

$$\begin{aligned} K_{F_i} + G_i' &\geq K_{F_i} + \ulcorner t\pi^*(K_X)|_{F_i}\urcorner + M_{5k_2+2}|_{F_i} \\ &\geq K_{F_i} + \ulcorner t\pi^*(K_X)|_{F_i}\urcorner + \sigma^*(2K_{F_0}) \end{aligned}$$

for all i. Furthermore, one can see that

$$\dim\Phi_{|K_{F_i}+\ulcorner t\pi^*(K_X)|_{F_i}\urcorner+\sigma^*(2K_{F_0})|}(F_i) \geq 1.$$

When $t = 0$, it is obvious. When $t > 0$, one need to use the vanishing theorem to prove it. Noting that the image of X' through ϕ_{t+7k_2+3} is irreducible and that

$$\dim\phi_{t+7k_2+3}(F_i) \geq 1$$

for all i, we can see $\dim\phi_{t+7k_2+3}(X) \geq 2$. The proof is complete. $\square$

Corollary 4.3. *Let $f : X \longrightarrow C$ be a derived family of surfaces from the m-canonical pencil $|mK_X|$ of a smooth projective 3-fold X of general type. Then*

(1) $q(X) \leq 1$ whenever $m \geq 82$.

(2) $q(X) = 0$ whenever $m \geq 143$.

Proof. According to [F] and Remark 6.6 in [Ko1], we always have $P_{20}(X) \geq 2$ if $q(X) > 0$. Let $k_2 = 20$ while applying Proposition 4.1 and Proposition 4.2, we get what we want. $\square$

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